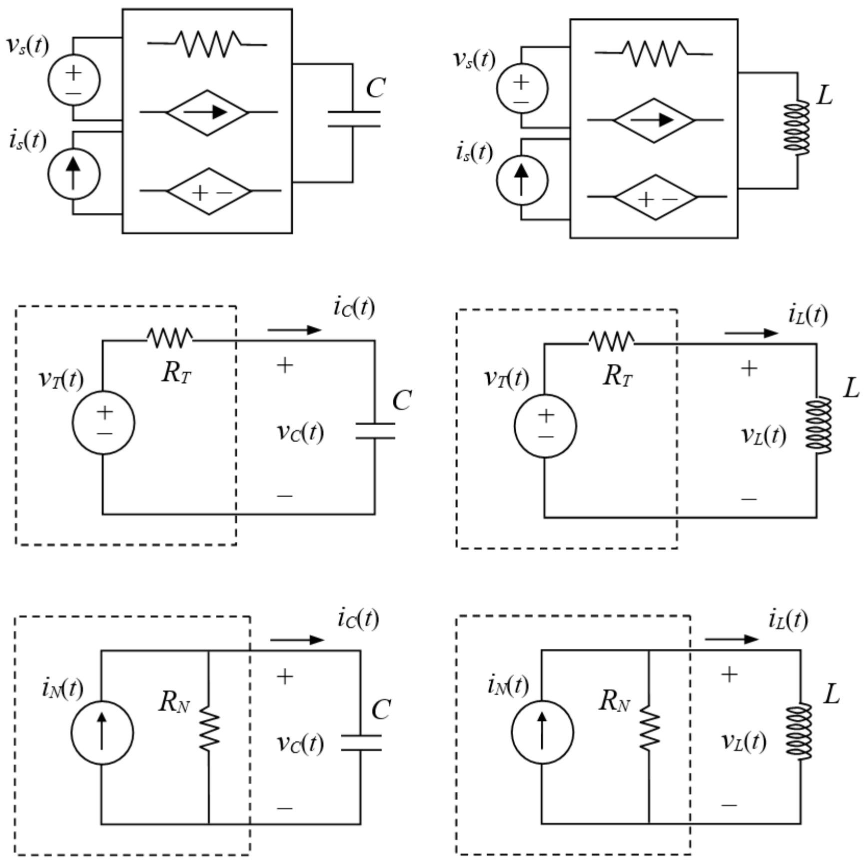
Chap 3 First-Order Linear Circuits



3.1 Laplace Transform

- Laplace transform of $f(t),\ t \ge 0$

(3.1-1) $\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$ or $\hat{f}(s) = \int_0^\infty f(t)e^{-st}dt$

- Laplace transform of $f'(t),\ t \ge 0$

(3.1-2) $\mathcal{L}\{f'(t)\} = s\hat{f}(s) - f(0)$

Prove: $\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}dt = \int_{t=0}^\infty e^{-st}df(t) = e^{-st}f(t)\Big|_0^\infty - \int_{t=0}^\infty f(t)de^{-st}$

$$= e^{-s\infty}f(\infty) - f(0) + s\int_{t=0}^\infty f(t)e^{-st}dt$$

If $Re(s) > 0$, we have $e^{-s\infty}f(\infty) = 0$ and thus

$$\mathcal{L}\{f'(t)\} = -f(0) + s\int_{t=0}^\infty f(t)e^{-st}dt = s\hat{f}(s) - f(0)$$

- Laplace transform of $\int_0^t f(\tau)d\tau,\ \ t \ge 0$

(3.1-3) $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\hat{f}(s)$

Prove: Let $g(t) = \int_0^t f(\tau)d\tau$, then $g(0)=0$ and $g'(t) = f(t)$.

From (3.1-7), we have $\mathcal{L}\{g'(t)\} = s\mathcal{L}\{g(t)\} - g(0) = s\mathcal{L}\{g(t)\}$

i.e., $\mathcal{L}\{g(t)\} = \frac{1}{s}\mathcal{L}\{g'(t)\}$ or $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{1}{s}\hat{f}(s)$

- Laplace transform of $e^{-at}f(t),\ \ t \ge 0$

(3.1-4) $\mathcal{L}\left\{e^{-at}f(t)\right\} = \int_0^\infty e^{-at}f(t)e^{-st}dt = \int_0^\infty f(t)e^{-(s+a)t}dt = \hat{f}(s+a)$

- $\mathcal{L}\{f(t)\}$ for $f(t)=1, t,\ \ e^{-at},\ \ e^{j\omega t},\ \ t \ge 0$.

(3.1-5) $\mathcal{L}\{1\} = \int_0^\infty 1\cdot e^{-st}dt = -\frac{1}{s}\int_0^\infty de^{-st} = -\frac{1}{s}e^{-st}\Big|_{t=0}^{\infty} = \frac{1}{s}$

(3.1-6) $\mathcal{L}\{t\} = \int_0^\infty t\cdot e^{-st}dt = -\frac{1}{s}\int_0^\infty t\cdot de^{-st} = -\frac{1}{s}\left(te^{-st}\Big|_0^\infty - \int_0^\infty e^{-st}dt\right) = \frac{1}{s^2}$

(3.1-7) $\mathcal{L}\left\{e^{-at}\right\} = \int_0^\infty e^{-at}e^{-st}dt = \int_0^\infty e^{-(s+a)t}dt = \frac{1}{s+a}$

(3.1-8) $\mathcal{L}\left\{e^{j\omega t}\right\} = \int_0^\infty e^{j\omega t}e^{-st}dt = \frac{1}{s-j\omega} = \frac{s}{s^2+\omega^2} + j\frac{\omega}{s^2+\omega^2}$

- Since $\mathcal{L}\left\{e^{j\omega t}\right\} = \mathcal{L}\{cos\,\omega t + j\,sin\,\omega t\}$ and $\mathcal{L}\left\{e^{-at}f(t)\right\} = \hat{f}(s+a)$, we have

(3.1-9) $\mathcal{L}\{cos\,\omega t\} = Re\left(\mathcal{L}\left\{e^{j\omega}\right\}\right) = \frac{s}{s^2+\omega^2}$

(3.1-10) $\mathcal{L}\{sin\,\omega t\} = Im\left(\mathcal{L}\left\{e^{j\omega}\right\}\right) = \frac{\omega}{s^2+\omega^2}$

(3.1-11) $\mathcal{L}\left\{te^{-at}\right\} = \frac{1}{(s+a)^2}$

(3.1-12) $\mathcal{L}\left\{e^{-at}\,cos\,\omega t\right\} = \frac{s+a}{(s+a)^2+\omega^2}$

(3.1-13) $\mathcal{L}\left\{e^{-at}\,sin\,\omega t\right\} = \frac{\omega}{(s+a)^2+\omega^2}$

3.2 RC Circuit

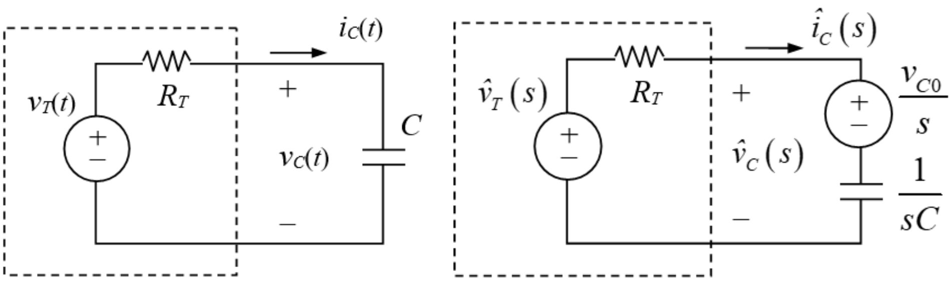
- Capacitor model with initial voltage $v_C(0) = v_{C0}$

(3.2-1) $$v_C(t) = v_{C0} + \frac{1}{C}\int_0^t i_C(\tau)\,d\tau$$

(3.2-2) $$\hat{v}_C(s) = \frac{v_{C0}}{s} + \frac{1}{C}\left(\frac{\hat{i}_C(s)}{s}\right) \;\Rightarrow\; \hat{v}_C(s) = \frac{v_{C0}}{s} + \underbrace{\frac{1}{sC}}_{z_c(s)}\hat{i}_C(s)$$

$z_C(s) = \dfrac{1}{sC}$: impedance of capacitor.

3.2.1 RC Circuit with Voltage Source



- Mathematic model

(3.2.1-1) $$\hat{v}_T(s) = R_T\,\hat{i}_C(s) + \frac{v_{C0}}{s} + \frac{1}{sC}\hat{i}_C(s) = \left(R_T + \frac{1}{sC}\right)\hat{i}_C(s) + \frac{v_{C0}}{s}$$

$$\Rightarrow\; \hat{i}_C(s) = \left(R_T + \frac{1}{sC}\right)^{-1}\left(\hat{v}_T(s) - \frac{v_{C0}}{s}\right) = \frac{sC}{1+sR_TC}\hat{v}_T(s) - \frac{Cv_{C0}}{1+sR_TC}$$

$$\Rightarrow\; \hat{v}_C(s) = \frac{v_{C0}}{s} + \frac{1}{sC}\hat{i}_C(s) = \frac{v_{C0}}{s+a} + \frac{a}{s+a}\hat{v}_T(s), \qquad a = \frac{1}{R_TC} > 0$$

$$\Rightarrow\; v_C(t) = \mathcal{L}^{-1}\left\{\frac{v_{C0}}{s+a}\right\} + \mathcal{L}^{-1}\left\{\frac{a}{s+a}\hat{v}_T(s)\right\} = v_{C0}e^{-at} + v_{Cp}(t)$$

- Neglect initial voltage v_{C0} as $t \to \infty$

(3.2.1-2) $$v_C(t) = \mathcal{L}^{-1}\left\{\frac{a}{s+a}\hat{v}_T(s)\right\} = v_{Cp}(t)$$

Since a>0, $v_{C0}e^{-at}$ will vanish as $t \to \infty$, and then $v_C(t) = v_{Cp}(t)$ depends only on the source $v_T(t)$.

- Constant voltage source $v_T(t) = V_T$ or $\hat{v}_T(s) = \frac{V_T}{s}$

(3.2.1-3) $\hat{v}_C(s) = \frac{v_{C0}}{s+a} + \frac{a}{s+a}\hat{v}_T(s) = \frac{v_{C0} - V_T}{s+a} + \frac{V_T}{s}$

$$\Rightarrow v_C(t) = (v_{C0} - V_T)e^{-at} + V_T = v_{C0}e^{-at} + V_T\left(1 - e^{-at}\right)$$

As *t* increases, $v_{C0}e^{-at}$ is gradually ignorable and finally $v_C(\infty) = V_T$ depends only on the source V_T.

- Sinusoidal voltage source $v_T(t) = cos\,\omega t$ or $\hat{v}_T(s) = \frac{s}{s^2 + \omega^2}$

(3.2.1-4) $\hat{v}_C(s) = \frac{v_{C0}}{s+a} + \frac{a}{s+a}\hat{v}_T(s) = \frac{v_{C0}}{s+a} + \frac{as}{(s+a)\left(s^2+\omega^2\right)}$

It can be obtained that

(3.2.1-5) $\frac{as}{(s+a)\left(s^2+\omega^2\right)} = \frac{\alpha}{s+a} + \frac{\beta s + \lambda\omega}{s^2+\omega^2}$

$$= \frac{(\alpha+\beta)s^2 + (\lambda\omega + a\beta)s + a\lambda\omega + \alpha\omega^2}{(s+a)\left(s^2+\omega^2\right)}$$

$$\Rightarrow \alpha + \beta = 0,\ \lambda\omega + a\beta = a,\ a\lambda\omega + \alpha\omega^2 = 0$$

$$\Rightarrow \alpha = -\frac{1}{1+(R_T C\omega)^2}.\ \ \beta = \frac{1}{1+(R_T C\omega)^2},\ \ \lambda = \frac{R_T C\omega}{1+(R_T C\omega)^2}$$

Hence, $\hat{v}_C(s) = \frac{v_{C0}+\alpha}{s+a} + \frac{\beta s + \lambda\omega}{s^2+\omega^2}$, i.e.,

(3.2.1-6) $v_C(t) = (v_{C0} + \alpha)e^{-at} + \beta\,cos\,\omega t + \lambda\,sin\,\omega t$

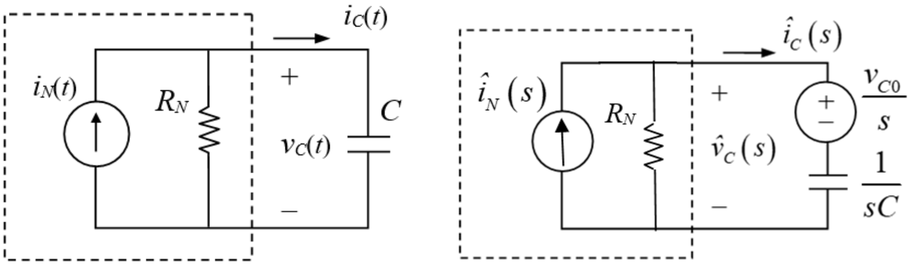
- As *t* increases, we have

(3.2.1-7) $v_C(t) = \beta\,cos\,\omega t + \lambda\,sin\,\omega t = \sqrt{\beta^2 + \lambda^2}\,cos(\omega t + \theta) = A\,cos(\omega t + \theta)$

where $A = \frac{1}{\sqrt{1+(R_T C\omega)^2}} < 1$ and $\theta = -tan^{-1}\left(\frac{\lambda}{\beta}\right) = -tan^{-1}(R_T C\omega) < 0$.

- A sinusoidal source $v_T(t) = cos\,\omega t$ results in $v_C(t) = A\,cos(\omega t + \theta)$, which is possessed of the same frequency ω, but with smaller magnitude A<1 and delay phase θ<0.

3.2.2 RC Circuit with Current Source



- Mathematic model

(3.2.2-1) $$i_N(t)=\frac{v_C(t)}{R_N}+i_C(t)=\frac{v_C(t)}{R_N}+C\,v_C'(t)$$

$$\Rightarrow\ v_C'(t)+\frac{1}{R_NC}v_C(t)=\frac{1}{C}i_N(t),\qquad v_C(0)=v_{C0}$$

Taking Laplace transform results in

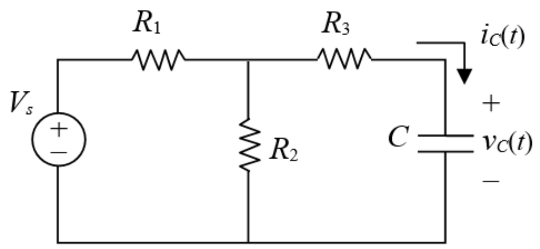
(3.2.2-2) $$s\,\hat{v}_C(s)-v_{C0}+\frac{1}{R_NC}\hat{v}_C(s)=\frac{1}{C}\hat{i}_N(s)$$

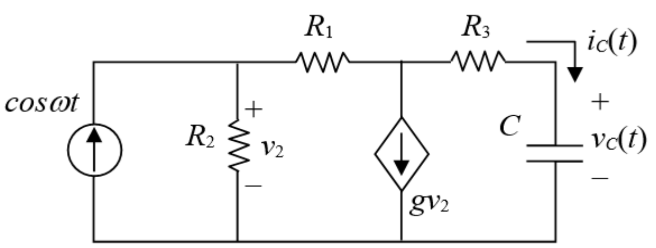
$$\Rightarrow\ (s+b)\hat{v}_C(s)=bR_N\hat{i}_N(s)+v_{C0}\qquad\left(b=\frac{1}{R_NC}\right)$$

$$\Rightarrow\ \hat{v}_C(s)=\frac{bR_N}{s+b}\hat{i}_N(s)+\frac{v_{C0}}{s+b}$$

$$\Rightarrow\ v_C(t)=\mathcal{L}^{-1}\left\{\frac{bR_N}{s+b}\hat{i}_N(s)\right\}+v_{C0}e^{-bt}$$

Example: Determine $v_C(t)$.





3.3 RL Circuit

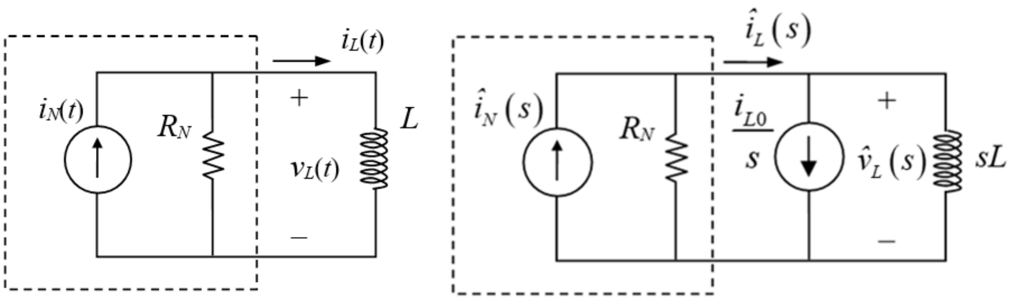
- Inductor model with initial current $i_L(0) = i_{L0}$

(3.3-1) $$i_L(t) = i_{L0} + \frac{1}{L}\int_0^t v_L(\tau)\,d\tau$$

(3.3-2) $$\hat{i}_L(s) = \frac{i_{L0}}{s} + \frac{1}{L}\left(\frac{\hat{v}_L(s)}{s}\right) \Rightarrow \hat{i}_L(s) = \frac{i_{L0}}{s} + \underbrace{\frac{1}{sL}}_{z_L^{-1}(s)}\hat{v}_L(s)$$

$z_L(s) = sL$: impedance of inductor

3.3.1 RL Circuit with Current Source



- Mathematic model

(3.3.1-1) $$\hat{i}_N(s) = \frac{\hat{v}_L(s)}{R_N} + \frac{i_{L0}}{s} + \frac{\hat{v}_L(s)}{sL} = \left(\frac{1}{R_N} + \frac{1}{sL}\right)\hat{v}_L(s) + \frac{i_{L0}}{s}$$

$$\Rightarrow \hat{v}_L(s) = \left(\frac{1}{R_N} + \frac{1}{sL}\right)^{-1}\left(\hat{i}_N(s) - \frac{i_{L0}}{s}\right) = \frac{sR_N L}{R_N + sL}\hat{i}_N(s) - \frac{R_N L i_{L0}}{R_N + sL}$$

$$\Rightarrow \hat{i}_L(s) = \frac{i_{L0}}{s} + \frac{1}{sL}\hat{v}_L(s) = \frac{i_{L0}}{s+a} + \frac{a}{s+a}\hat{i}_N(s), \qquad a = \frac{R_N}{L}$$

$$\Rightarrow i_L(t) = \mathcal{L}^{-1}\left\{\frac{i_{L0}}{s+a}\right\} + \mathcal{L}^{-1}\left\{\frac{a}{s+a}\hat{i}_N(s)\right\} = i_{L0}e^{-at} + i_{Lp}(t)$$

- Neglect initial current i_{L0} as $t \to \infty$

(3.3.1-2) $$i_L(t) = \mathcal{L}^{-1}\left\{\frac{a}{s+a}\hat{i}_N(s)\right\} = i_{Lp}(t)$$

Since a>0, $i_{L0}e^{-at}$ will vanish as $t \to \infty$, and then $i_L(t) = i_{Lp}(t)$ depends only on the source $v_T(t)$.

- Constant current source $i_N(t)=I_N$ or $\hat{i}_N(s)=\dfrac{I_N}{s}$

(3.3.1-3) $\hat{i}_L(s)=\dfrac{i_{L0}}{s+a}+\dfrac{a}{s+a}\hat{i}_N(s)=\dfrac{i_{L0}-I_N}{s+a}+\dfrac{I_N}{s}$

$$\Rightarrow\ i_L(t)=(i_{L0}-I_N)e^{-at}+I_N=i_{L0}e^{-at}+I_N\left(1-e^{-at}\right)$$

As t increases, $i_{L0}e^{-at}$ is gradually ignorable and finally $i_L(\infty)=I_N$ depends only on the source I_N

- Sinusoidal voltage source $i_N(t)=sin\,\omega t$ or $\hat{i}_N(s)=\dfrac{\omega}{s^2+\omega^2}$

(3.3.1-4) $\hat{i}_L(s)=\dfrac{i_{L0}}{s+a}+\dfrac{a}{s+a}\hat{i}_N(s)=\dfrac{i_{L0}}{s+a}+\dfrac{a\omega}{(s+a)\left(s^2+\omega^2\right)}$

It can be obtained that

(3.3.1-5) $\dfrac{a\omega}{(s+a)\left(s^2+\omega^2\right)}=\dfrac{\alpha}{s+a}+\dfrac{\beta s+\lambda\omega}{s^2+\omega^2}$

$$=\frac{(\alpha+\beta)s^2+(\lambda\omega+a\beta)s+a\lambda\omega+\alpha\omega^2}{(s+a)\left(s^2+\omega^2\right)}$$

$$\Rightarrow\ \alpha+\beta=0,\ \lambda\omega+a\beta=0,\ a\lambda\omega+\alpha\omega^2=a\omega$$

$$\Rightarrow\ \alpha=\frac{L\omega/R_N}{1+\left(L\omega/R_N\right)^2},\ \beta=-\frac{L\omega/R_N}{1+\left(L\omega/R_N\right)^2},\ \lambda=\frac{1}{1+\left(L\omega/R_N\right)^2}.$$

Hence, $\hat{i}_L(s)=\dfrac{i_{L0}+\alpha}{s+a}+\dfrac{\beta s+\lambda\omega}{s^2+\omega^2}$, i.e.,

(3.3.1-6) $i_L(t)=(i_{L0}+\alpha)e^{-at}+\beta\cos\omega t+\lambda\sin\omega t$

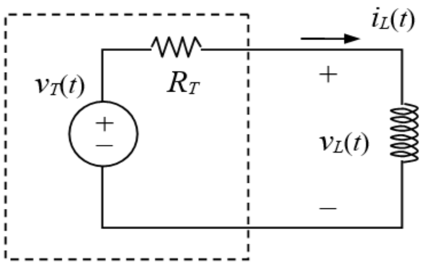
- As t increases, we have

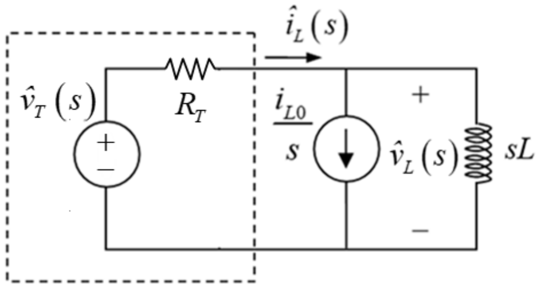
(3.3.1-7) $i_L(t)=\beta\cos\omega t+\lambda\sin\omega t=\sqrt{\beta^2+\lambda^2}\,sin(\omega t+\theta)=A\,sin(\omega t+\theta)$

where $A=\dfrac{1}{\sqrt{1+\left(L\omega/R_N\right)^2}}<1$ and $\theta=tan^{-1}\left(\dfrac{\beta}{\lambda}\right)=-tan^{-1}\left(L\omega/R_N\right)<0$.

- A sinusoidal source $i_N(t)=sin\,\omega t$ results in $i_L(t)=A\,sin(\omega t+\theta)$, which is possessed of the same frequency ω, but with smaller magnitude A<1 and delay phase θ<0.

3.3.2 RL Circuit with Voltage Source





- Mathematic model

(3.3.2-1) $v_T(t) = R_T i_L(t) + v_L(t) = R_T i_L(t) + L i_L'(t)$

$$\Rightarrow\ i_L'(t) + \frac{R_T}{L} i_L(t) = \frac{1}{L} v_T(t), \qquad i_L(0) = i_{L0}$$

Taking Laplace transform results in

(3.3.2-2) $s\hat{i}_L(s) - i_{L0} + \frac{R_T}{L}\hat{i}_L(s) = \frac{1}{L}\hat{v}_T(s)$

$$\Rightarrow\ (s+b)\hat{i}_L(s) = \frac{1}{L}\hat{v}_T(s) + i_{L0} = \frac{b}{R_T}\hat{v}_T(s) + i_{L0} \qquad \left(b = \frac{R_T}{L}\right)$$

$$\Rightarrow\ \hat{i}_L(s) = \frac{b/R_T}{s+b}\hat{v}_T(s) + \frac{i_{L0}}{s+b}$$

$$\Rightarrow\ i_L(t) = \mathcal{L}^{-1}\left\{\frac{b/R_T}{s+b}\hat{v}_T(s)\right\} + i_{L0}e^{-bt}$$

<u>Example</u>: Determine $i_L(t)$.